B.Sc./6th Sem (H)/MATH/23(CBCS)

2023

6th Semester Examination

MATHEMATICS (Honours)

Paper : C 14-T

[Ring Theory and Linear Algebra-II]

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

Answer any ***ten*** questions : 2×10=20

1. Let $Z_i = \{a + ib : a, b \in Z\}$ be the ring of Gaussian integers. Show that $1 + i$ is irreducible element in Z_i.

2. Show that the polynomial $f(x) = x^2 + \overline{3}x + \overline{2} \in \mathbb{Z}_6[x]$ has four zeros in $\mathbb{Z}_6$.

3. Let W_1 and W_2 be subspaces of a finite dimensional inner product space. Prove that $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

4. Let T be a linear operator on a finite dimensional inner

P.T.O.

product space V. If T has an eigenvector, then its adjoint operator T^* does have so.

5. Let R be a UFD and $a, b, c \in R\setminus\{0\}$ such that $a|bc$ and $gcd\,(a, b) \sim 1$. Then prove that $a|c$.

6. State ***Eisenstein's criterion*** for irreducibility of a polynomial $f(x) \in \mathbb{Z}[x]$ over $\mathbb{Z}$.

7. Let U be a subset of a vector space V over the field F. Prove that the annihilator of U (denoted by U^0) is a subspace of the dual space V^*.

8. Show that the polynomial $f(x) = 21x^3 - 3x^2 + 2x + 9$ is irreducible over $\mathbb{Q}$.

9. For any linear transformation T, define the adjoint linear transformation T^* of T. Hence prove that

$$(T_1T_2)^* = T_2^*T_1^*$$

10. Let P be the linear operator on the vector space $\mathbb{R}^2$ over $\mathbb{R}$ defined by $P(x, y) = (x, 0)$ for all $(x, y) \in \mathbb{R}^2$. Find the minimal polynomial for P.

11. If T is a unitary linear transformation, then show that the characteristic roots of T all have absolute value 1.

12. Consider the vector space $\mathbb{R}^2$ over $\mathbb{R}$ equipped with

the standard inner product. Consider $\alpha = (1,2)$ and $\beta = (-1,1)$. Find an element $\gamma \in \mathbb{R}^2$ for which $<\alpha, \gamma> = -1$ and $<\beta, \gamma> = 3$.

13. Let V be an inner product space and $x, y \in V$. If $<x, v> = <y, v>$ for all $v \in V$ then prove that $x = y$.

14. Let V be the vector space $\mathbb{C}^2$ over $\mathbb{C}$ with the standard inner product. Let T be the linear operator on V defined by $T(1, 0) = (1, -2)$ and $T(0, 1) = (i, -1)$. Find $T^*(\alpha)$ where $\alpha = (x_1, x_2) \in V$.

15. Give an example of a 2×2 complex matrix A such that A^2 is normal but A is not normal.

Group - B

Answer any *four* questions : $5 \times 4 = 20$

16. (i) Let R be an integral domain and $f(x), g(x) \in R[x]$. Prove that

$$\deg(f(x)g(x)) = \deg f(x) + \deg g(x)$$

(ii) Find all the associates of $x^2 + [2]$ in $\mathbb{Z}_7[x]$.

3+2=5

17. (i) Is the ring of all 2×2 matrices with their entries from $\mathbb{Z}$ a PID? Justify your answer.

P.T.O.

(ii) Exhibit (with proper justification) an ideal I in the polynomial ring $\mathbb{Z}[x]$ so that I is not a principal ideal. 3+2=5

18. Let V be the vector space of all polynomials over $\mathbb{R}$ with degree less than or equal to 2. Let t_1, t_2, t_3 be three distinct real numbers and L_1, L_2, L_3 be three linear functionals on V defined by $L_i(p) = p(t_i)$ for all $i = 1,2,3$. Find the basis $\mathcal{B} = \{p_1, p_2, p_3\}$ of V such that $\{L_1, L_2, L_3\}$ becomes the dual basis of $\mathcal{B}^*$ of $\mathcal{B}$.

19. Consider the matrix $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$ over the field of real numbers. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

20. Let V be a finite dimensional vector space over the field F and T be a linear operator on V. If T is diagonalizable, then prove that the minimal polynomial for T is a product of distinct linear factors.

21. Let V be a finite dimensional inner product space and T be an invertible linear operator on V. Then show that

(i) T^* is also an invertible operator on V.

(ii) $(T^*)^{-1} = (T^{-1})^*$. 2+3=5

Group - C

Answer any *two* questions : 10×2=20

22. (i) Check whether the following statement is true or false : "Let R be an integral domain, $a, b \in R \setminus \{0\}$ and $d = \gcd(a,b)$. Then there exist $x, y \in R$ such that $d = ax + by$." Give proper justification in support of your answer. 4

(ii) Let V be a vector space of dimension m over a field F and W be a vector subspace of dimension k of V where $1 \le k < n$. Then prove that the dimension of the subspace $\{f | f \in V^*,\ f(w) = 0 \forall w \in W\}$ is $n - k$. 4

(iii) Let V be a complex or real inner product space. Then show that the induced norm $\| \ \|$ satisfies the following equality :

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + 2\|y\|^2\right)$$

(***Parallelogram Law***) for all $x, y \in V$. 2

23. (i) Construct a field of 8 elements. 5

(ii) Let V be the vector space of all polynomial functions R to R of degree ≤ 2. Let t_1, t_2, t_3 be three distinct real numbers and let $L_i : V \to F$ be

P.T.O.

such that $L_i(p(x)) = p(t_i)$, $i = 1,2,3$. Show that $\{L_1, L_2, L_3\}$ is a basis of $\hat{V}$. Determine a basis of V such that $\{L_1, L_2, L_3\}$ is its dual. 5

24. (i) Prove that $I = < x^2 + 1 >$ is a prime ideal in $\mathbb{Z}[x]$ but not a maximal ideal in $\mathbb{Z}[x]$. 5

(ii) Let W be the plane in $\mathbb{R}^3$ spanned by the set $S = \{(1,2,2), (-1,0,2)\}$. Then find an orthonormal basis $\mathcal{B}'$ for W applying Gram-Schmidt orthogonalization process and extend $\mathcal{B}'$ to an orthonormal basis $\mathcal{B}$ of V. 3+2=5

25. (i) For any prime p, show that the p^{th} cyclotomic polynomial is irreducible over $\mathbb{Q}$.

(ii) Let V be an inner product space and $S = \{v_1, v_2, \ldots\ldots, v_n\}$ be an orthonormal subset of V. Show that for any $x \in V$, $\|x\|^2 \geq \sum_{i=1}^{n} |< x, v_i >|^2$. 4+6

2023

6th Semester Examination

MATHEMATICS (Honours)

Paper : DSE 3-T

[CBCS]

Full Marks : 60 Time : Three Hours

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[Mechanics]

Group - A

Answer any **ten** questions : 2×10=20

1. An artificial satellite revolves about the earth at a height H above the surface. Find the orbital speed so that a man in the satellite will be in a state of weightlessness.

2. Define 'apse' of a central orbit. Show that, at an apse, a particle is moving at right angles to the radius vector.

3. When the equilibrium of a rigid body under the action of a number of coplanar forces will be stable or unstable in nature?

P.T.O.

4. Find the C.G. of a uniform arc of a circle.

5. Show that the centres of suspension and oscillation of a compound pendulum are interchangeable.

6. Define equi-momental bodies. Write the conditions for equi-momental bodies.

7. Find moment of inertia of circular ring of mass M and of radius 'a' about any diameter.

8. Define compound pendulum. What do you mean by simple-equivalent pendulum?

9. State conservation of linear momentum under finite forces. Also state conservation of energy.

10. Prove that the momental ellipsoid at the centre of the elliptic plate whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\left(\frac{1}{a^2}+\frac{1}{b^2}\right)z^2 = \text{constant}.$$

11. A particle describes an ellipse about a focus and when at the end of minor axis receives a small impulse towards that focus which communicates a velocity u to the particle. Show that the eccentricity is increased by $ua\left(1-e^2\right)^{3/2}/h$.

12. A particle describes an ellipse under a force $\frac{\mu}{(\text{distance})^2}$ towards the focus; if it was projected with velocity v from a point at a distance r from the centre of force, show that its periodic time is $\frac{2\pi}{\mu}\left[\frac{2}{r}-\frac{v^2}{\mu}\right]^{-3/2}$.

13. Find the velocity of an artificial satellite of the earth, given $g = 9.8$ m/sec^2, radius of earth $= 6.4 \times 10^8$/metres. (Assume that the satellite is moving very close to the surface of the earth).

14. The position of a particle of mass m moving in space referred to a set of rectangular axes at any instant t is $\left(a\cos nt, a\sin nt, \frac{1}{2}at^2\right)$. Find the magnitude and direction of the acceleration.

15. What is meant by principal axes of a given material system at a point? State the condition so that a given straight line may be a principal axis of the material system at any point of its length.

Group - B

Answer any ***four*** questions : $5\times4=20$

16. If a system of forces in one plane reduces to a couple whose moments is G and when each force is turned through a right angle it reduces to a couple H. Prove that when each force is turned through an angle α, the system is equivalent to a couple whose moment is $G\cos\alpha + H\sin\alpha$.

17. Find the C.G. of area enclosed by the curves $y^2 = ax$ and $x^2 + y^2 = 2ax$ lying in the first quadrant.

18. A particle is projected at right angle to the line joining it to a centre of force, attracting according to the law of inverse square of the distance, with a velocity $\frac{\sqrt{3}}{2}V$, where V denotes the velocity from infinity. Find the eccentricity of the orbit described and show that the periodic time is $2\pi T$, where T is the time taken to describe the major-axis of the orbit with velocity V.

19. A particle of unit mass is projected with velocity u at an inclination α above the horizon in a medium whose resistance is k-times the velocity. Show that the direction of the path described will again make an angle α with the horizon after a time $\frac{1}{k}\log\left(1+\frac{2ku}{g}\sin\alpha\right)$.

20. OA, OB, OC are the edges of a cube of side a and OO', AA', BB', CC' are its diagonals; along OB', $O'A$, BC, $C'A'$ act forces equal to P, $2P$, $3P$, $4P$; show that they are equivalent to force $\sqrt{35}P$ at O along a line whose direction cosines are proportional to $-3, -5, 6$ together with a couple $\frac{Pa}{2}\sqrt{114}$ about a line whose direction cosines are proportional to $7, -2, 2$.

21. Two equal uniform rods, AB and AC, are freely hinged at A and rest in a straight line on a smooth table. A blow is struck at B perpendicular to the rods, show that the kinetic energy generated is $\frac{7}{4}$ times what it would be if the rods were rigidly fastened together at A.

Group - C

Answer any *two* questions : 10×2=20

22. (i) A beam of length l rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizon are α and β, and the centre of gravity of the beam divides it in the ratio $a : b$. Find the position of equilibrium of the beam and show that the equilibrium is unstable. 6

P.T.O.

(ii) If a hemisphere rests in equilibrium with its curved surface in contact with a rough plane inclined to a horizontal at an angle θ then show that the inclination of the plane of the hemisphere to the horizontal is $\sin^{-1}\left(\frac{8}{3}\sin\theta\right)$, provided $\theta < \sin^{-1}\frac{3}{8}$. 4

23. (i) Prove that every given system of forces acting on a rigid body can be reduced to a wrench. 5

(ii) Six forces, each equal to P, act along the edges of a cube, taken in order which do not meet a given diagonal. Show that their resultant is a couple of moment $2\sqrt{3}Pa$, where a is the edge of the cube. 5

24. (i) Find the kinetic energy of a body moving in two dimensions. 4

(ii) A lamina in the form of an ellipse is rotating in its own plane about one of its foci with angular velocity ω. This focus is set free and the other, at the same instant is fixed, show that the ellipse now rotate about it with angular velocity $\omega\frac{2-5e^2}{2+3e^2}$. 6

25. (i) Having given the moments and products of inertia of a rigid body about three perpendicular concurrent axes. Find the moment of inertia of the body about an axis, with known direction cosines through that

point. Hence deduce the equation of momental ellipsoid of the body at that point. 3+2

(ii) A rough uniform rod of length $2a$ is placed on a rough table at right angles to its edge. If its C.G. be initially at a distance b beyond the edge, show that the rod will begin to slide when it has turned through an angle $\tan^{-1}\left(\frac{\mu a^2}{a^2+9b^2}\right)$, where μ is the coefficient of friction. 5

3

P.T.O.

B.Sc./6th Sem (H)/MATH/23(CBCS)

2023

6th Semester Examination

MATHEMATICS (Honours)

Paper : C 13-T

[Metric Spaces and Complex Analysis]

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

Answer any ***ten*** questions : 2×10=20

1. State the Banach fixed point theorem.

2. What do you mean by complete metric space?

3. Define uniform continuity.

4. Write down the Heine-Borel property.

5. Let (X, d) be a metric space in which A and B are two intersecting connected sets. Show that $A \cup B$ is connected.

P.T.O.

6. Let $f(z) = \dfrac{|z|}{\operatorname{Re}(z)}$ if $\operatorname{Re}(z) \neq 0$

$\qquad\quad = 0$ if $\operatorname{Re}(z) = 0$

Show that $f(z)$ is not continuous at $z = 0$.

7. Show that the function $u = \cos x \cosh y$ is harmonic.

8. Find the radius of convergence $\sum_{h=2}^{\infty} \dfrac{z^n}{n.(\log n)^2}$.

9. Evaluate $\oint_C \dfrac{1}{z} dz$, $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$.

10. Show that $u(x, y) = 4xy - x^3 + 3xy^2$, is harmonic.

11. Show that a convergent sequence in a metric space is bounded.

12. Give an example, in the real time $\mathbb{R}$, of the sequence $\{x_n\}$ such that $|x_n - x_{n+1}| \to 0$ (as $n \to \infty$) but $\{x_n\}$ is not Cauchy.

13. Show that for any subset A of a metric space (X, d), the function $f : X \to \mathbb{R}$ given by $f(x) = d(x, A)$, $x \in X$, is uniformly continuous.

14. Show that $\lim_{z \to z_0} f(z)g(z) = 0$ if $\lim_{z \to z_0} f(z) = 0$ and if there exists a positive integer M such that $|g(z)| \leq M$ for all z in some neighbourhood of z_0.

15. Let $T(z)=\dfrac{az+b}{cz+d}$ be a bilinear transformation. Show that ∞ is a fixed point of T if and only if $c=0$.

Group - B

Answer any ***four*** questions : 5×4=20

16. Show that continuous image of a compact metric space is compact.

17. Check whether the function is differentiable at $z=0$. Also check whether it satisfies C-R equations.

$$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}} \quad (z\neq 0)$$
$$= 0 \quad (z=0)$$

18. If $f(z)$ is an alytic function within and on a closed contour C, and if a is any point within C, then show that

$$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz.$$

19. (i) Prove that a metric space (X, d) having the property that every continuous map $f:X\rightarrow X$ has a fixed point, is connected. 2

(ii) Let (X, d) be a complete metric space and $T:X\rightarrow X$ be a contraction on X. Then for $x\in X$, show that the sequence $\{T^n(x)\}$ is a convergent sequence. 3

P.T.O.

20. (i) Determine whether the set $S=\{(x, y):0<x\leq 1, x^2+y^2=4\}$ is compact in $\mathbb{R}^2$. 3

(ii) Let X be an infinite set endowed with the discrete metric. Show that every infinite subset of (X, d) is bounded but not totally bounded. 2

21. Evaluate :

(i) $\int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}dz$ where C is the circle $|z|=3$ described in the positive sence. 2

(ii) $\int_C \frac{zdz}{(9-z^2)(z+i)}$ where C is the circle $|z|=2$ described in the positive sence. 3

Group - C

Answer any *two* questions : 10×2=20

22. (i) State and prove Liouville's theorem.

(ii) If $f(z)$ is a regular analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2.$$

23. (i) Show that any compact subset of a metric space is closed and bounded.

(ii) Show that two metrics d_1, d_2 on a set X are equivalent iff the identity map $I_X:(X, d_1)\to(X, d_2)$ is a homomorphism.

24. (i) Show that the map $f:[0, 1]\to[0, 1]$, defined by

$$f(x)=x-\frac{x^2}{2},\quad x\in[0, 1]$$ is a weak contraction but not a contraction map. 3

(ii) Let (X, d) be a complete metric space and $f:X\to X$ be a contraction map with Lipschitz constant t $(0<t<1)$. If $x_0\in X$ is the unique fixed point of f, show that $d(x,x_0)\le\frac{1}{1-t}d(x,f(x))$, for all $x\in X$. 5

(iii) Show that a contraction of a bounded plane set may have the same diameter as the set itself. 2

25. (i) Let $f(z)=u(x,y)+iv(x,y)$, $z=x+iy$ and $z_0=x_0+iy_0$. Let the function f be defined in a domain D except possibly at the point z_0 in D. Then prove that $\lim\limits_{z\to t_0} f(z)=u_0+iv_0$ if and only if

$\lim\limits_{x\to x_0} u(x,y)=u_0$ and $\lim\limits_{y\to y_0} v(x,y)=v_0$. 5

(ii) Show that when $0<|z|<4$,

$$\frac{1}{4z-z^2}=\frac{1}{4z}+\sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}.$$ 5

B.Sc./6th Sem (H)/MATH/23(CBCS)

2023

6th Semester Examination

MATHEMATICS (Honours)

Paper : DSE 4-T

[CBCS]

Full Marks : 60 Time : Three Hours

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[Mathematical Modelling]

Group - A

Answer any ***ten*** questions : 2×10=20

1. What is Monte Carlo simulation?

2. Use the middle-square method to generate 2 random numbers considering the seed $x_0 = 3043$.

3. Find $L^{-1}\left\{\dfrac{s+2}{s^2(s+3)}\right\}$.

4. Show that Laplace transform of the function $f(t)=t^n$, $-1<n<0$, exists, but it is not piecewise continuous on every finite subinterval in the range $t \geq 0$.

P.T.O.

5. The cost of any non-basic variable can be reduced without limit, without affecting the optimal basic feasible solution to the LPP. Justify.

6. If $L\{f(t)\} = \dfrac{50s+3}{s^4+3s^2+(k-4)s}$ and $\lim\limits_{t\to\infty} f(t) = 1$ then find the value of k where k is a constant.

7. Let $f(t)$ be the continuous function on $[0, \infty]$ whose Laplace Transform exists. If $f(t)$ satisfies $\int_0^t (1-\cos(t-u))\, f(u)\, du = t^4$ then find $f(t)$.

8. Is the solution $\left(1, \frac{1}{2}, 0, 0, 0\right)$ a basic solution of the equations

$$x_1 + 2x_2 + x_3 + x_4 = 2$$

$$x_1 + 2x_2 + \frac{1}{2}x_3 + x_5 = 2\,?$$

9. What is meant by singularity of a linear ordinary differential equation?

10. Write the general expression of $P_n(z)$.

11. What do you mean cycling in linear congruence?

12. Prove that $L^{-1}\left\{\dfrac{f(s)}{s^2}\right\} = \int_0^t \int_0^v F(u)\, du\, dv$.

13. Does the Laplace transform of $\frac{\cos at}{t}$ exist?

14. What is a probabilistic process?

15. Why we generate random number?

Group - B

Answer any *four* questions : 5×4=20

16. How do you generate random numbers between 0 and 1 that follow uniform distribution using linear congruence method?

17. Using Monte Carlo simulation, write an algorithm to calculate that part of the volume of an ellipsoid

$$\frac{x^2}{2}+\frac{y^2}{4}+\frac{z^2}{8}\leq 16$$

18. In the LPP

Maximize $Z = 3x_1 + 5x_2$

Subject to $x_1 + x_2 \leq 1$

$2x_1 + 3x_2 \leq 1$

$x_1, x_2 \geq 0,$

obtain the variation of $c_j\,(j=1,2)$ without changing the optimality of solution.

P.T.O.

19. Find $L^{-1}\left\{\frac{1}{\sqrt{p}(p-a)}\right\}$ by the convolution integral.

20. Generate 15 random numbers using middle-square method taking $x_0 = 3043$.

21. Prove the Final value theorem $\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, sF(s)$.

Group - C

Answer any *two* questions : 10×2=20

22. (i) Use Laplace transform to solve the following initial-value problem

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = h(t),\ y(0) = 0,\ y'(0) = 0 \text{ where}$$

$$h(t) = \begin{cases} 2, & 0 < t < 4 \\ 0, & t > 4 \end{cases}$$

(ii) Prove that $L\{\sinh at \cos at\} = \frac{a\left(s^2 - 2a^2\right)}{s^4 + 4a^4}$. 7+3

23. (i) Write a Monte Carlo simulation algorithm for a harbour with unloading facilities for ships finding the answers of the following questions :

1. What is the average and maximum times per ship in the harbour?

2. What are the average and maximum waiting times per ship?

3. What percentage of the time are the unloading facilities idle?

(ii) What are the disadvantages of linear congruence method to generate random numbers? 7+3

24. (i) Find the Laplace inverse of the function $\frac{1}{(p+a)^3}$.

(ii) Solve the following LPP using simplex method

Maximize $Z = 6x + 4y$

subject to $-x + y \leq 12$

$x + y \leq 24$

$2x + 5y \leq 80$

$x, y \geq 0.$ 3+7

25. Find the solution of the Bessel differential equation of order λ at the neighbourhood of $x = 0$. Discuss the case when $\lambda = 0$. 8+2

$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \lambda^2) y$

P.T.O.